Stereo Imaging and Geolocation

Stereo imaging serves two purposes. The first is to produce three-dimensional topographic images. These can be used to produce digital elevation models (DEM) and to improve the identification of items of interest. The second is to locate observable points on the ground with respect to a reference geoid (an elliptical surface approximating the mean surface of the Earth). The latter is referred to as geolocation and is the focus of this section. Stereo imaging requires a scene to be viewed from two directions. This can be accomplished with a satellite in one of three ways: with a single camera viewing the scene on different orbit passes (SPOT 1); with a single camera on the same orbit pass by first viewing forward and then viewing aft (IKONOS II); or using a pair of cameras, one viewing forward and the other viewing aft (Corona KH 4B and SPOT 5). These options are depicted in Figure 1.

Figure 1. Stereo Imaging Options

The first option (Figure 1a) can produce stereo strip maps but requires cloud free conditions for orbit passes separated days apart. Also the LOS zenith angles of the two images are likely to be unequal, which implies different ground sample distances. The second option (Figure 1b) allows the two images to be collected close in time, can have equal zenith angles, but cannot produce long strip maps of stereo imagery. The third option (Figure 1c) is ideal from the standpoint of stereo imaging. The stereo pairs are collected close in time, the image pairs have equal zenith angles, and long strip maps can be produced.

Geolocation Problem

Stereo imaging is most easily understood when the images are collected using a frame camera. Then all the pixels in an image block are collected at an instant in time from a specific vantage point. And all the pixels in an image block maintain a fixed registration to one another. The disadvantages of using a frame camera are (1) it must be operated in a step-stare fashion, and (2) it is more difficult to produce wide image swaths.
For these reasons, virtually all satellite stereo imaging is performed using pushbroom line scanning, in which the pixels in an image strip are collected line by line at different instances in time. And line wander is another issue that must be dealt with.

To simplify the presentation we will use the model of a frame camera with the understanding that when imaging a point rather than an extended scene that there is no difference. Consider a stereo image pair created from two overlaid images from sensor positions \( S_1 \) and \( S_2 \). The two images are not necessarily taken from the same orbit or even from the same satellite. The geolocation problem is to locate an observable ground point \( G \) with respect to a reference geoid given the location of points \( S_1 \) and \( S_2 \), and knowledge of the LOS directions to \( G \).

Neither the positions of \( S_1 \) and \( S_2 \) nor their LOS directions are known precisely. Both bias and random errors come into play. Bias errors can be largely removed by imaging over a test range with one or more ground control points (GCP). A GCP is an observable object which location is known a priori, say by using a GPS receiver. Then with the appropriate calibration to remove the bias errors one is left to deal with the random errors. Geolocation accuracy is normally specified in terms of a 90% circular error probability (CE 90) in the local horizontal plane, and a 90% linear elevation error probability (LE 90) along the local vertical.

**Geometry for the General Case.** Referring to Figure 2, \( \mathbf{r}_1 \) and \( \mathbf{r}_2 \) are the LOS vectors from the two sensors to an observable ground point \( G \), and \( \mathbf{n} \) is the vector normal to the \( \mathbf{r}_1 \mathbf{r}_2 \) plane (shown in blue) defined by

\[
\mathbf{n} = \frac{\mathbf{r}_1 \times \mathbf{r}_2}{r_1 r_2}
\]

The \( X_G Y_G Z_G \) Cartesian triad with origin at \( G \) has \( Z_G \) along the local vertical, and \( X_G \) axis normal to the plane defined by unit vector \( \mathbf{n} \) and the local vertical \( Z_G \). \( Y_G \) completes the triad.

![Figure 2. Geometry in the General Case](image)

Given that neither the positions of \( S_1 \) and \( S_2 \) nor the directions of the lines of sight are not known precisely, there will be errors in predicting the location of \( G \). Given the standard
deviations for these predictions along the X_G and Y_G axes we can determine the circular error probability. Two approximations for CE 90 are:

\[
(CE \ 90)_1 = 2.146 \sqrt{\sigma_{x_G} \sigma_{y_G}} \quad \quad (CE \ 90)_2 = 2.146 \sqrt{\frac{\sigma_{x_G}^2 + \sigma_{y_G}^2}{2}}
\]

The first definition uses the geometric mean of the two horizontal standard deviations, while the second definition uses the square root of the average variances in the two horizontal directions. The two definitions yield nearly the same results over the zenith angle range of interest. The elevation probability LE 90 depends only on the Z_G axis error:

\[
LE \ 90 = 1.645 \sigma_{z_G}
\]

The difficulty in computing the CE 90 and LE 90 values in the general case is that the geolocation errors are logically formulated in the x_Gy_Gz_G frame rather than the X_GY_GZ_G frame. While the x_G coincides with X_G, y_G is along n. The required transformation is

\[
\begin{bmatrix}
y_G \\ z_G
\end{bmatrix} = \begin{bmatrix}
\cos \alpha & \sin \alpha \\
-\sin \alpha & \cos \alpha
\end{bmatrix} \begin{bmatrix}
Y_G \\ Z_G
\end{bmatrix}
\]

where 90° - \( \alpha \) is the angle between Z_G and n.

Let \( \Theta \) be the convergence angle between the vectors \( r_1 \) and \( r_2 \), \( \psi_1 \) be the zenith angle between \( r_1 \) and Z_G, and \( \psi_2 \) be the zenith angle between \( r_2 \) and Z_G. Within the \( r_1r_2 \) plane, the angles \( \vartheta_1 \) and \( \vartheta_2 \) locate \( r_1 \) and \( r_2 \) with respect to the \( z_G \) axis. The angle between the two viewing directions, measured about the local vertical \( Z_G \) is \( \beta \). The angle \( \alpha \) is the minimum angle between \( Z_G \) and the \( r_1r_2 \) plane. Given \( \psi_1 \), \( \psi_2 \) and \( \beta \) we can determine the convergence angle \( \Theta \) from the expression

\[
\cos \Theta = \cos \psi_1 \cos \psi_2 + \sin \psi_1 \sin \psi_2 \cos \beta
\]

To determine \( \vartheta_1 \), \( \vartheta_2 \) and \( \alpha \) first solve for the corner angle \( \Psi_2 \) in Figure 2 using

\[
\cos \Psi_2 = \frac{\cos \psi_2 - \cos \psi_1 \cos \Theta}{\sin \psi_1 \sin \Theta}
\]

where \( \Theta = \vartheta_1 + \vartheta_2 \). Then compute

\[
\sin \alpha = \sin \psi_1 \sin \Psi_2, \quad \cos \vartheta_1 = \cos \psi_1 / \cos \alpha, \quad \cos \vartheta_2 = \cos \psi_2 / \cos \alpha
\]

**Position and LOS Knowledge Errors.** Let \( p_1 \) and \( p_2 \) denote the satellite position uncertainties normal to the two lines of sight, and let \( \varepsilon_1 \) and \( \varepsilon_2 \), denote the LOS uncertainties as shown in Figure 3. The red G point denotes the predicted location relative to the true position in black.
The corresponding triangulation errors for $G$ are

$$
\Delta x_G = \left[ (p_1 + r_1\epsilon_1) \frac{\tan \theta_2}{\cos \theta_1} + (p_2 + r_2\epsilon_2) \right] \cdot \frac{1}{\tan \theta_1 + \tan \theta_2}
$$

$$
\Delta z_G = \left[ \frac{(p_1 + r_1\epsilon_1)}{\cos \theta_1} - \frac{(p_2 + r_2\epsilon_2)}{\cos \theta_2} \right] \cdot \frac{1}{\tan \theta_1 + \tan \theta_2}
$$

There are two independent measurements of the $y$ error component:

$$
\Delta y'_G = (p_1 + r_1\epsilon_1) \quad \text{and} \quad \Delta y''_G = (p_2 + r_2\epsilon_2)
$$

For unbiased random errors with $p_1 = p_2 = p$ and $\epsilon_1 = \epsilon_2 = \epsilon$, the one-sigma errors are

$$
\sigma_{x_G} = \frac{1}{\tan \theta_1 + \tan \theta_2} \sqrt{\left( \sigma_p^2 + r_1^2 \sigma_{LOS}^2 \right) \cdot \left( \frac{\tan \theta_2}{\cos \theta_1} \right)^2 + \left( \sigma_p^2 + r_2^2 \sigma_{LOS}^2 \right) \cdot \left( \frac{\tan \theta_1}{\cos \theta_2} \right)^2}
$$

$$
\sigma_{z_G} = \frac{1}{\tan \theta_1 + \tan \theta_2} \sqrt{\left( \sigma_p^2 + r_1^2 \sigma_{LOS}^2 \right) \cdot \left( \frac{1}{\cos \theta_1} \right)^2 + \left( \sigma_p^2 + r_2^2 \sigma_{LOS}^2 \right) \cdot \left( \frac{1}{\cos \theta_2} \right)^2}
$$

For random position and LOS errors normal to the $r_1r_2$ plane the out of plane location error (along the $y_G$ axis) is

$$
\sigma_{y_G} = \frac{\sqrt{\sigma_p^2 + r_1^2 \sigma_{LOS}^2} \cdot \sqrt{\sigma_p^2 + r_2^2 \sigma_{LOS}^2}}{\sqrt{2\sigma_p^2 + (r_1^2 + r_2^2)\sigma_{LOS}^2}}
$$

using the maximum likelihood estimator form.

We thus have the geolocation errors in $x_G, y_G, z_G$ coordinates. The last step is to compute them in the local vertical $X_G, Y_G, Z_G$ coordinates, which involves the rotation $\alpha$ about the $x_G$ axis:

$$
\sigma_{x_o} = \sigma_{x_G}
$$

$$
\sigma_{y_o} = \sqrt{\sigma_{y_G}^2 \cos^2 \alpha + \sigma_{z_G}^2 \sin^2 \alpha}
$$

$$
\sigma_{z_o} = \sqrt{\sigma_{y_G}^2 \sin^2 \alpha + \sigma_{z_G}^2 \cos^2 \alpha}
$$
Symmetrical Viewing. This is the special case when \( r_1 = r_2 = r \) and \( \vartheta_1 = \vartheta_2 = \vartheta \). The errors along the \( x_G_y_Gz_G \) axes reduce to

\[
\sigma_{x_G} = \sqrt{\frac{\sigma_p^2 + r^2 \sigma_{LOS}^2}{2 \cos \vartheta}}, \quad \sigma_{y_G} = \sqrt{\frac{\sigma_p^2 + r^2 \sigma_{LOS}^2}{2}}, \quad \sigma_{z_G} = \sqrt{\frac{\sigma_p^2 + r^2 \sigma_{LOS}^2}{2 \sin \vartheta}}
\]

Resolving along the \( X_GY_GZ_G \) we obtain

\[
\sigma_{x_G} = \frac{\sqrt{\sigma_p^2 + r^2 \sigma_{LOS}^2}}{\sqrt{2 \cos \vartheta}}
\]

\[
\sigma_{y_G} = \frac{\sqrt{\sigma_p^2 + r^2 \sigma_{LOS}^2}}{\sqrt{2}} \sqrt{\cos^2 \alpha + \frac{\sin^2 \alpha}{\sin^2 \vartheta}}
\]

\[
\sigma_{z_G} = \frac{\sqrt{\sigma_p^2 + r^2 \sigma_{LOS}^2}}{\sqrt{2}} \sqrt{\sin^2 \alpha + \frac{\cos^2 \alpha}{\sin^2 \vartheta}}
\]

When \( G \) lies nominally in the \( r_1r_2 \) plane, the angle \( \alpha \) is zero and the \( x_Gy_Gz_G \) and \( X_GY_GZ_G \) frames coincide. In this instance \( \vartheta \) and \( \psi \) are synonymous.
**Rotation about the LOS.** LOS pointing knowledge is much more critical than knowledge of the rotation about the LOS. Consider a frame camera with N-by-N pixels (Figure 4). For a rotation $\theta$ about the center of the array, the translation of the outermost pixels along the x and y axes is $(N/2)p\theta$ where $p$ is the pixel pitch. In units of pixels the displacement error is $(N/2)\theta$. For an 8K by 8K array ($N = 8000$) with an allowable pixel displacement of $1/4$ pixel, the allowable rotation uncertainty is $1/16000$ radians or 62.5 $\mu$rad. The required LOS pointing requirements are typically an order of magnitude tighter.

![Figure 4. Rotation about the LOS](image)

Rotation pointing knowledge about the LOS is somewhat greater problem for line scanners simply because $N$ is typically a much larger number than for frame cameras.

**Measurement Synchronization.** Typical integration times vary between 1 and 3 ms. For a satellite velocity on the order of 7 km/s, this means that the satellite position will change between 7 and 21 m over the integration time, which is a large number compared to the $< 1$ m accuracy to which a GPS can determine satellite position. Nevertheless the differences between taking the measurements at the beginning, middle or end of the integration interval is unimportant so long as the position and LOS measurements correspond to the exact same instant.

**Extension to Line Scan Sensors.** Unlike a frame camera that collects an entire image block at an instant in time, a line scan camera collects only one line at a time, so there is not a single perspective center for an image block. It also means that the LOS rate must be controlled to within tight tolerances of the orbit rate in order to maintain the same relative geometry for each line in an image block as depicted in Figure 4. Then the above equations for geolocation also apply to a line scan sensor with the understanding that the satellite position and LOS knowledge must be continuously updated for each image line.

The actual measurements need not be made every line, but must be frequent enough to ensure accurate interpolation for lines in-between measured lines. Interpolation of satellite position between GPS measurements is straightforward because the dynamic model is simple. Assuming a circular orbit, the desired LOS rate is a constant, in which case LOS interpolation is also be straightforward. However, a real system is not completely immune from line wander, in
which case the post processing compensation techniques (previously discussed under pushbroom scan) can provide a well-registered stereo image block.

IKONOS II Example. Table 1 presents error prediction values for the IKONOS II satellite in its 682-km orbit collecting a stereo image by first looking forward and then looking aft. It is assumed that the fore and aft zenith angles are equal. The results are equivalent to having a pair of stereo cameras and varying the off nadir view angles. The error probabilities are based on a 1-m satellite random position error normal to the LOS and a 6-μr random LOS knowledge error. The resulting target error normal to the LOS is given by

\[ \sigma_N = \sqrt{\sigma_p^2 + r^2\sigma_{LOS}^2} \]

that increases with zenith angle because of the increase in slant range. The horizontal and vertical standard deviations are plotted as a function of the convergence angle in Figure 6. These results are in close agreement with the results presented by Grodecki, Dial and (Ref 7) for the same position and pointing uncertainties using the Rational Polynomial Camera (RPC) model for IKONOS II. The corresponding 90% error probabilities are plotted in Figure 7 as a function of the zenith angle (half the convergence angle).
TABLE 1. Error Functions as a Function of Zenith Angle for the IKONOS II Orbit.

<table>
<thead>
<tr>
<th>Orbit Parameters</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Complete orbits per day, ( I )</td>
<td>14</td>
</tr>
<tr>
<td>Coverage repeat period, ( N )</td>
<td>140</td>
</tr>
<tr>
<td>Residual, ( K )</td>
<td>89</td>
</tr>
<tr>
<td>Orbits per solar day, ( Q = I + (K/N) )</td>
<td>14.636</td>
</tr>
<tr>
<td>Orbits per repeat period, ( R = NQ )</td>
<td>2049</td>
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<tr>
<td>Orbit period, ( \text{min} )</td>
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<tr>
<td>Mean orbit rate, ( n )</td>
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<td>Semi-major axis, ( a )</td>
<td>7060</td>
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<tr>
<td>Orbit altitude</td>
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<tr>
<td>Sun synchronous inclination</td>
<td>98.127</td>
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<table>
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<tr>
<th>Viewing Geometry</th>
<th>Value</th>
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<tr>
<td>Earth centered viewing offset, ( \varphi )</td>
<td>4.52</td>
</tr>
<tr>
<td>Range, ( r )</td>
<td>684.0</td>
</tr>
<tr>
<td>Off nadir view angle, ( \varphi )</td>
<td>4.52</td>
</tr>
<tr>
<td>Zenith angle, ( \psi )</td>
<td>5.0</td>
</tr>
<tr>
<td>Convergence angle, ( \Theta )</td>
<td>10</td>
</tr>
<tr>
<td>Time between stereo looks</td>
<td>15.9</td>
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<th>Knowledge Errors</th>
<th>Value</th>
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<td>Position error normal to LOS, ( \sigma_x )</td>
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<tr>
<td>LOS error, ( \sigma_{\text{LOS}} )</td>
<td>6.0</td>
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<tr>
<td>Target error normal to LOS, ( \sigma_N )</td>
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<table>
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<tr>
<th>Stereo Errors</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Horizontal error, ( \sqrt{\sigma_x, \sigma_y} )</td>
<td>3.0</td>
</tr>
<tr>
<td>Vertical error, ( \sigma_z )</td>
<td>34.3</td>
</tr>
<tr>
<td>CE 90 (geometric mean definition)</td>
<td>6.4</td>
</tr>
<tr>
<td>LE 90</td>
<td>56.4</td>
</tr>
</tbody>
</table>

Figure 6. Standard Deviations
Figure 7. Error Probabilities
A Priori Ground Elevation Information.

With a priori knowledge of the target elevation information it is possible to estimate the local horizontal coordinates of the target with a single look (that is without a stereo image pair). This is depicted in the left hand portion of Figure 8 where $\sigma_N$ is the target uncertainty normal to the LOS due to satellite position and LOS uncertainties, and $\sigma_{hap}$ is the a priori uncertainty in the ground elevation. The shaded area denotes the uncertainty area in the $X_G Z_G$ plane. In this case we have

$$\sigma_{X_G} = \sqrt{\sigma_N^2 + \sigma_{hap}^2 \sin^2 \psi}, \quad \sigma_{Z_G} = \sigma_{hap}$$

The right hand portion of Figure 8 shows two looks (stereo image pair) along with a priori elevation data. The horizontal error is unaffected by the elevation information, while the vertical error is the maximum likelihood estimate using triangulation and the a priori information:

$$\sigma_{X_G} = \frac{\sigma_N}{\sqrt{2 \cos \psi}}, \quad \sigma_{Z_G} = \sqrt{\frac{\sigma_N^2 \sigma_{hap}^2}{\sigma_N^2 + 2 \sigma_{hap}^2 \sin^2 \psi}}$$

Figure 8. A priori Ground Elevation Information

Image Blocks.

For a frame camera an image block is logically an image frame. For a line scanner, an image block is a sequence of lines the form the equivalence of an image frame. Assume that observable objects can be registered to within 1/3 of a pixel relative to the image block. If the image block is ideal (no distortions) than the accuracy to which any point can be located with respect to the reference geoid is the combination of the geolocation prediction and the pixel registration error. The main sources of block distortion are image distortion of the optics, and line wander for a line scanner.
**Optical Distortion.** Depending on the optical design of the telescope, image distortion (Figure 9) may or may not be a serious concern. In general it must be calibrated so that its effects can be compensated in post processing.

![Unaberrated, Barrel Distortion (contraction), Pin Cushion Distortion (expansion)](image)

*Figure 9. Optical Distortion*

**Line Wander.** Pushbroom imaging is its susceptibility to line-wander, a low frequency phenomena. A typical image block can be collected within one or two seconds. This may seem short, but not so short that oscillations with frequencies of only a few Hertz can play havoc. Typical line wander signatures are depicted in Figure 10.

![Cross track line wander, Along track line wander, See-Saw line wander](image)

*Figure 10. Line Wander Signatures.*

With proper design of the imaging arrays (previously discussed) these errors can be largely removed with post processing. The residual error grows slowly with the number of lines in a block. With this approach it appears possible to limit the one-sigma registration error over a 20,000-line block to less than a pixel.

**Use of a GCP.** A GCP can be used to improve the geolocation accuracy of all the points in the block over that obtained solely by triangulation of stereo images. Let $\sigma_{GCP}$ denote the position accuracy...
normal to the LOS to which the GPC is known (a priori) and registered within an image block. The improved estimates for the target error normal to the LOS, denoted by primes, is given by

$$
\sigma'_N = \sqrt{\sigma_N^2 \cdot \sigma_{GCP}^2}
$$

Two GCPs in an image block are required to remove errors due to rotation about the LOS. However, this is not usually a major issue because the relative insensitivity to rotation about the LOS, as previously discussed.

Figure 11 shows the improvement gained by the presence of a single GCP in a stereo image block collected by the IKONOS II satellite over a test range in Mississippi (Grodecki, J. and Dial, G., “IKONOS Geometric Accuracy Validation”, Proceedings of the ISPRS Commission I Midterm Symposium, Denver, CO, Nov 2002). The vertical errors are in red and the horizontal errors are in black with the scale of the errors denoted in meters. The errors on the left are those without the GCP. Those on the right are with the GCP. The reason for the apparent bias in residual horizontal errors with the GCP is unstated; though it is most likely due to uncompensated line wander of the pushbroom sensors.

![a) without a GCP](image1.png)  ![b) with a GCP](image2.png)

Figure 11. Horizontal (black) and Vertical (red) Errors for a Stereo Imaging Block Collected by the IKONOS Satellite

**Block Adjustment Using Tie Points**

Consider the case when there are partially overlapping stereo image blocks each formed by two by fore and aft image. Assume that in the overlap region there is an observable point, or tie point (TP). Unlike a GCP, there is no a priori knowledge of the TP location. Being observable a TP can be used to “stitch” adjacent blocks of imagery to improve the accuracy over that achieved using stand-alone image blocks. This process of enhancing accuracy by stitching is called “block adjustment”. There are two factors that affect the “relative” accuracy of a TP: the 1/3 of a pixel registration error for any observable
point, and the residual errors in the image block grid (e.g. residual line wander after post processing compensation). Thus a relative TP accuracy of 1 m is reasonable for a line scan system.

There are a number of papers on the subject of block adjustment. A most informative one is by three individuals at Space Imaging (now GEOEYE), who have extensively modeled the stereo imagery produced by the IKONOS satellite: Grodecki, J., Dial, G. and Lutes, J., *Error Propagation In Block Adjustment of High-Resolution Satellite Images*, ASPRS 2003 Annual Conference Proceedings, May 2003. They employ a Rational Polynomial Camera (RPC) model for block adjustment.

Two examples of their results are shown in Figure 12. In each case there are four partly overlapping side-by-side stereo image blocks with three TPs in each of the block overlapping areas. The zenith angles employed were 20° and there were no internal block distortion errors. The stereo viewing is in the N-S direction (top to bottom) so the block overlaps correspond to an adjacent orbit passes. The outer gold circles represent the horizontal error predictions using satellite position and LOS knowledge estimates without tie points. The red ovals show the predicted improvement when TP measurements are included. In this example the improvements are approximately a factor of two in the along track direction and slightly better in the cross track direction.

In the second example a GCP is added in the left stereo image block. The residual errors (red ovals) when both the TPs and the GCP are included are reduced by about another factor of two (compared to the red ovals in the first case) for the left stereo pair. The errors grow slightly for blocks farther from the one with the GCP.

Figure 12. Stereo Block Adjustment Error Propagation Using RPC Model.
**Simple Model.** The following simple model appears to capture the salient features of the more robust and accurate RPC model, and tends to agree with the above results. Consider the case of two partially overlapping stereo image blocks as depicted in Figure 13. There is a TP in the block overlap region and there may be a GCP in the left stereo block.

![Figure 13. Two Partially Overlapping Image Blocks with a Tie Point](image)

This model can be represented by the two-ball analogy in Figure 14. The problem is to estimate the positions \(c_1\) and \(c_2\) given the measurements \(m_1\) of \(c_1\), \(m_2\) of \(c_2\) and \(m_{21}\) of \(c_2 - c_1\). The measurements have random errors with variances \(\sigma_1^2\), \(\sigma_2^2\) and \(\sigma_{21}^2\), respectively.

![Figure 14. Simple Two-Ball Problem](image)

The measurement equation is simply

\[
\mathbf{m} = \mathbf{A}\mathbf{c}
\]

where \(\mathbf{m}\) represents the measurements. The maximum likelihood estimates for \(\mathbf{c}\) are given by

\[
\hat{\mathbf{c}} = \left(\mathbf{A}^\top \Lambda^{-1} \mathbf{A}\right)^{-1} \mathbf{A}^\top \Lambda^{-1} \mathbf{m}
\]

The covariance matrix is \(\mathbf{C} = \left(\mathbf{A}^\top \Lambda^{-1} \mathbf{A}\right)^{-1}\) where \(\Lambda\) are the variances:
\( \Lambda = \begin{bmatrix} \sigma_1^2 & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & 0 & \sigma_{12}^2 \end{bmatrix} \)

Expanding we obtain

\[
C = \frac{1}{\sigma_1^2 + \sigma_2^2 + \sigma_{21}^2} \begin{bmatrix}
(\sigma_2^2 + \sigma_{21}^2)\sigma_1^2 & -\sigma_1^2\sigma_2^2 \\
-\sigma_1^2\sigma_2^2 & (\sigma_1^2 + \sigma_{21}^2)\sigma_2^2
\end{bmatrix}
\]

In applying this analogy to geolocation, the position errors for the \( i \)th block are given by

\[
\sigma_{X_{gi}} = \frac{\sqrt{C_{ii}}}{\sqrt{2}} \cdot \cos \psi, \quad \sigma_{Y_{gi}} = \frac{\sqrt{C_{ii}}}{\sqrt{2}}, \quad \sigma_{Z_{gi}} = \frac{\sqrt{C_{ii}}}{\sqrt{2}} \cdot \sin \psi
\]

where the \( C_{ii} \) are the diagonal elements of the \( C \) matrix. And the 90% error probabilities are:

\[
CE\ 90 = 2.146 \sqrt{\sigma_{X_{gi}} \sigma_{Y_{gi}}} \quad LE\ 90 = 1.645 \sigma_{Z_{gi}}
\]

If there were no difference measurements (equivalent to setting \( \sigma_{21} = \infty \)), the errors for the two balls would be simply \( \sigma_1 \) and \( \sigma_2 \), respectively. If difference measurement error were zero, the errors for the two balls would both be equal to \( \sigma_1 \sigma_2 / \sqrt{\sigma_1^2 + \sigma_2^2} \). If \( \sigma_1 = \sigma_2 \), the error is reduced by the \( \sqrt{2} \).

Consider two cases with \( \psi = 20^\circ \). In the first case there is no coupling and the target measurement errors normal to the LOS are \( \sigma_1 = \sigma_2 = 3 \) m and there is no coupling (\( \sigma_{21} = \infty \)). Then each block has the same values:

\[
\sigma_{X_0} = 2.60 \text{ m}, \quad \sigma_{Y_0} = 2.52 \text{ m}, \quad \sigma_{Z_0} = 4.31 \text{ m}
\]

\[
CE\ 90 = 4.70 \text{ m}, \quad LE\ 90 = 10.20 \text{ m}
\]

If a 1-m TP is added (\( \sigma_{21} = 1 \) m) the block values drop to:

\[
CE\ 90 = 3.41 \text{ m} \quad LE\ 90 = 7.40 \text{ m}
\]

So simply by using image blocks and tie points it is possible to improve geolocation accuracy. Indeed accuracy should increase as the number of blocks is increased.

**Extension to Four Blocks.** Consider the four-block geometries in Figure 14. The geometry on the right is that cited by Grodecki, et al. But to the accuracy of this model, the following analysis also applies to the staggered array on the left. The latter is directly applicable to parallel, partially overlapping pushbroom scans. Note that because pushbroom scans do not overlap in the along track directions, the image strips must be partitioned into staggered image blocks to implement block adjustment.
The measurement matrix for this case is

\[
\mathbf{m} = \mathbf{Ac}
\]

\[
\begin{bmatrix}
m_1 \\
m_2 \\
m_3 \\
m_4 \\
m_{21} \\
m_{32} \\
m_{43}
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 1 & 0 & 0 \\
0 & -1 & 1 & 0 \\
0 & 0 & -1 & 1
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
c_3 \\
c_4
\end{bmatrix}
\]

from which

\[
\mathbf{D} = \mathbf{A}^T \mathbf{A}^{-1} =
\begin{bmatrix}
1/\sigma_1^2 + 1/\sigma_{21}^2 & -1/\sigma_{21}^2 & 0 & 0 \\
-1/\sigma_{21}^2 & 1/\sigma_2^2 + 1/\sigma_{21}^2 + 1/\sigma_{32}^2 & -1/\sigma_{32}^2 & 0 \\
0 & -1/\sigma_{32}^2 & 1/\sigma_3^2 + 1/\sigma_{32}^2 + 1/\sigma_{43}^2 & -1/\sigma_{43}^2 \\
0 & 0 & -1/\sigma_{43}^2 & 1/\sigma_4^2 + 1/\sigma_{43}^2
\end{bmatrix}
\]

With four blocks diagonal elements of the inverse \( \mathbf{C} = (\mathbf{A}^T \mathbf{A}^{-1})^{-1} \) are most conveniently determined numerically.

A series of cases using this model are presented in Table 2. In all the cases the target measurement error \( \sigma_N \) (combination of satellite position error and LOS error) is 3 m. In Case 1 there is block coupling through TPs with 1-m errors. The improvement due to block coupling is approximately 40%.
Case 2 is the same as Case 1, but with a GCP added to block 1 with an assumed accuracy of 1 m. The block 1 CE is now only 30% of that in Case 1 while the block 4 CE is about 50%. Finally in Case 3, there is a GCP in both Blocks 1 and 4. These are the same trends reported by Grodecki, Dial and Lutes in the previously cited paper using the RPC model.

**TABLE 2. Ensemble Block Adjustment**

<table>
<thead>
<tr>
<th>Case 1</th>
<th>1-m TPs added</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Block 1</td>
</tr>
<tr>
<td>σX</td>
<td>1.30</td>
</tr>
<tr>
<td>σY</td>
<td>1.23</td>
</tr>
<tr>
<td>σZ</td>
<td>3.58</td>
</tr>
<tr>
<td>CE90</td>
<td>2.71</td>
</tr>
<tr>
<td>LE90</td>
<td>5.89</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Case 2</th>
<th>1-m TPs and 1-m GCP in Block 1</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Block 1</td>
</tr>
<tr>
<td>σX</td>
<td>0.68</td>
</tr>
<tr>
<td>σY</td>
<td>0.64</td>
</tr>
<tr>
<td>σZ</td>
<td>1.87</td>
</tr>
<tr>
<td>CE90</td>
<td>1.42</td>
</tr>
<tr>
<td>LE90</td>
<td>3.08</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Case 3</th>
<th>1-m TPs and 1-m GCP in Blocks 1 &amp; 4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Block 1</td>
</tr>
<tr>
<td>σX</td>
<td>0.65</td>
</tr>
<tr>
<td>σY</td>
<td>0.61</td>
</tr>
<tr>
<td>σZ</td>
<td>1.79</td>
</tr>
<tr>
<td>CE90</td>
<td>1.36</td>
</tr>
<tr>
<td>LE90</td>
<td>2.95</td>
</tr>
</tbody>
</table>

Previously we indicated that geolocation accuracy improves with the addition of TPs independent of the availability of GCPs. This is illustrated in Table 4 for increasing number of ensemble blocks linked by 1-m accuracy TPs. In the limit when the TP errors approach zero, the block errors approach the limit of the block measurement error divided by the square root of the number of blocks.

**TABLE 4. Multiple Blocks Coupled by TPs.**

<table>
<thead>
<tr>
<th></th>
<th>Block 1</th>
<th>Block 2</th>
<th>Block 3</th>
<th>Block 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>CE 90</td>
<td>4.70</td>
<td>3.41</td>
<td>2.93</td>
<td>2.71</td>
</tr>
<tr>
<td></td>
<td>3.41</td>
<td>2.81</td>
<td>2.52</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2.93</td>
<td>2.52</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2.71</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>LE 90</td>
<td>10.20</td>
<td>7.40</td>
<td>6.37</td>
<td>5.89</td>
</tr>
<tr>
<td></td>
<td>7.40</td>
<td>6.10</td>
<td>5.48</td>
<td></td>
</tr>
<tr>
<td></td>
<td>6.37</td>
<td>5.48</td>
<td>5.89</td>
<td></td>
</tr>
<tr>
<td></td>
<td>5.89</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
All this suggests that an ensemble block adjustment over large areas (Figure 14) with multiple TPs, with or w/o GCPs, can achieve ever increasing geolocation accuracies with the increase in the number of overlapping blocks.

Figure 14. Extension to Large Areas (9-block example).

**Error Flow Summary.** At this point there may be some confusion as to the overall error model, especially for a line scanner. Figure 15 is introduced to aid in the understanding. The single point errors refer to the direct measurements of a particular point in the stereo image. The Instrument errors refer to the Kalman filter outputs of the gyro and star sensor measurements for the specific image line for the point of interest. The single point error also includes the unmeasured factors due to dynamic and thermal misalignment of the camera perspective center with respect to the measurement axis system.

The GCP error has three components. The first is the accuracy to which the GCP is known a priori. Only the components normal to the LOS have relevance. This is reasonably taken to be 1 m. The second is the accuracy to which it can be registered within the image. A third of a pixel is deemed reasonable. Because the GCP and the image point are not collocated, one must also include factors contributing to block distortion. The main factor is the line wander error between the GCP line and the point of interest line. A 1-pixel error is reasonable limit for block distortion. The single block error for the point of interest is computed using a maximum likelihood estimate for the combined measurement and GCP errors.

The TP errors normal to the LOS come into play for ensemble block adjustment. The TP error tree is the same as a GCP except that there is no a priori error. The ensemble block adjustment is then performed using a RPC model or for a rough estimate using the simplified model above.
Figure 15. Error Flow
References

Between 2001 and 2004 there were at least ten papers published by Space Imaging personnel on stereo imaging as applied to the IKONOS satellite. Six are cited below:


